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Two-dimensional scaling in magnetic systems with non-integer characteristic lengths

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Abstract. We studied the ferromagnetic Ising model on two-dimensional finite systems with non-integer characteristic lengths. First we considered very long strips with a finite number I of complete rows and one partially filled row with probability x, which is the two-dimensional version of a layer-by-layer growth. At fixed temperature, when the characteristic length $L = I + x$ increases, the free energy per spin, the specific heat and the magnetic susceptibility oscillate, attaining relative extremes at integer L. The oscillations in the free energy are interpreted as surface corrections related to the oscillations in the mean coordination number. Finite-size scaling relations are not satisfied with continuous L, but still hold for fixed x and variable I, where the differences of mean widths are integers. For fixed x, the fits of the free energy give the conformal anomaly $c = \frac{1}{2}$ with very good accuracy. We also studied strips with discretized Gaussian distributions of widths, with non-integer means L and rms deviation $\Delta L = 1$. In these structures, the thermodynamic quantities vary monotonically with continuous L, but some methods for calculating critical exponents do not work properly when generalized to continuous L. We also obtain $c = \frac{1}{2}$ with good accuracy in these systems. We discuss the possible implications of our results to real systems behaviour.

1. Introduction

In the study of magnetic systems with finite dimensions, one of the important problems is the dependence of physical quantities on the characteristic length of the structure. Finite-size scaling theories connect this problem to the critical behaviour of the corresponding infinite systems and provide some tools which are widely used in the study of critical phenomena [1, 2]. The possibility of applications of those theories increased with the developments in the experimental techniques to produce and analyse nanostructures such as thin films, small magnetic clusters or islands and ferromagnetic strips [3–5]. These applications, however, are frequently limited by simplifications in the geometry of the structures which are theoretically studied. The latter systems usually have uniform lengths (integer lengths in units of the lattice parameter) but, in experiments, the characteristic lengths are generally means over certain distributions of lengths. Some modern techniques may lead to almost layer-by-layer growth of thin films, which favours the approximation of real systems behaviour by theoretical models. However, with those techniques it is also possible to construct films with a finite number of (almost) complete layers and one partially filled layer, whose non-integer thicknesses are measured with good accuracy [6, 7]. Thus we are confronted with the questions of how the physical properties depend on the characteristic length of the structure

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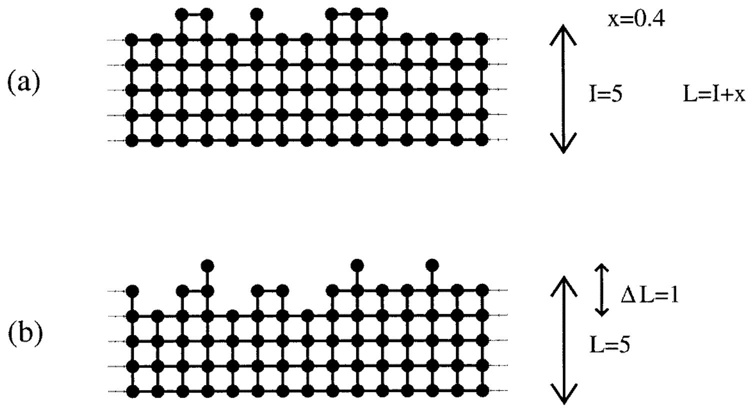

Figure 1. (*a*) Long strip with a finite number $I = 5$ of complete rows and one partially filled row with probability $x = 0.4$. (*b*) Strip with a Gaussian distribution of widths, with mean $L = 5$ and rms deviation $\Delta L = 1$. Full circles represent the spins and full lines represent bonds between nearest neighbours.

when it is not integer, and how they depend on the distribution of microscopic lengths around a mean value.

Although the behaviour of several magnetic models in structures with uniform characteristic lengths in one, two and three dimensions has already been widely analysed, little is known about the effects of a non-uniform distribution of lengths. A step in that direction was taken in recent papers, where we studied the Ising model on strips [8] and thin films [9, 10] with Gaussian distributions of thicknesses. However, the mean thicknesses L were always integers in those systems. The roughness patterns were defined by the relations between the rms deviation of thicknesses ΔL and the length L. It was shown that finite-size scaling relations were satisfied in those systems, and their corrections due to different roughness patterns were discussed.

In this paper we also consider two-dimensional systems with strip geometry. This geometry has attracted much interest in the study of the two-dimensional critical behaviour of pure [2, 11, 12] and disordered systems [13, 14]. Recently it was also considered in a theoretical model [15] to describe experiments on Fe strips deposited on W(110) [5]. Here we study the ferromagnetic Ising model in strips with non-integer characteristic lengths L, grown under different conditions, in order to analyse the dependence of the physical properties on those lengths. First, we consider the two-dimensional version of a perfect layer-by-layer growth: infinitely long strips with a finite number I of complete rows and one partially filled row with probability x (figure 1(*a*)). These strips have characteristic lengths $L = I + x$ (width I with probability $1 - x$, width $I + 1$ with probability x). Subsequently we extend the study of strips with Gaussian distributions of widths (figure 1(*b*)) to non-integer mean widths L. Hereafter we refer to these systems as Gaussian strips.

From the theoretical point of view, this paper is relevant as a study of the extensions of finite-size scaling to non-integer lengths. Although the results presented here cannot be directly related to experiments, they provide important information on the relations of growth conditions and finite-size scaling when the lengths of the structures vary continuously. For instance, in the strips with one incomplete row we show that the free energy per spin and their second derivatives (specific heat and magnetic susceptibility) oscillate as L increases, at fixed temperature, but the same does not occur with the Gaussian strips. This is an effect of the particular ways of filling the different layers in the two processes, which lead to oscillations of the mean coordination number in the strips with one incomplete row. We will also show that some finite-size scaling techniques for calculating critical exponents, which are usually applied to integer lengths L, do not work properly when generalized to continuous values of L, but are still valid if the values of L differ by integers. Finally, we will analyse the finite-size corrections to the free energy and show that the presence of random boundaries still leads to the value of the central charge (conformal anomaly) $c = \frac{1}{2}$.

We study strips with one incomplete row up to $L = 14$, and Gaussian strips up to $L = 11$ with a fixed rms deviation $\Delta L = 1$. We use transfer matrix techniques [11, 12] to calculate the free energy per spin of very long strips, and obtain their specific heats and magnetic susceptibilities from numerical derivatives of f. Most calculations are done at the critical temperature T_c of the two-dimensional Ising ferromagnet ($k_B T_c / J = 2.269\ldots$) [16], where those quantities scale in particularly interesting forms.

This paper is organized as follows. In section 2 we present details of the calculations and analyse the free energy data of strips with a single incomplete row. In section 3 we analyse the specific heat and magnetic susceptibility of those strips. In section 4 we present results in the Gaussian strips. In section 5 we calculate the surface free energy and the conformal anomaly in both systems. In section 6 we summarize our results and conclusions.

2. Free energy of strips with one incomplete row

We have studied strips of length $N = 10^5$ sites, with complete rows of widths I between 3 and 13 (figure 1(*a*)). The incomplete row was filled with probability x ranging from 0.1 to 0.9, at intervals of 0.1, and with $x = 0.02$ and $x = 0.98$. Free boundary conditions were considered. The coupling $J > 0$ was constant for all nearest-neighbour pairs. The external magnetic field h was zero. For a certain strip length ($L = I + x$), the values of physical quantities presented below are averages over four estimates, each obtained from a different realization.

The free energy is calculated using standard transfer matrix techniques [11,12]. The total free energy is

$$F(T, h) = -N \ln \lambda_L^0 \tag{1}$$

where λ_L^0 is the largest eigenvalue of the transfer matrix. The free energy per site is $f(T, h) = F(T, h)/(NL)$.

In figure 2(*a*) we show the total free energy per column $F(T_c)/N$ of those strips as a function of the characteristic length L, near $L = 6$. In figure 2(*b*) we show the free energy per site $f(T_c)$ for all values of L analysed. For any L, the fluctuations among the estimates in the four realizations are much smaller than the size of the points, thus error bars are not shown.

It is clear that the derivative of F with respect to L is discontinuous at integer L (figure 2(*a*)), and that it leads to the oscillations of f (figure 2(*b*)). F always decreases with L at fixed temperature, since the internal energy decreases and the entropy increases when the total number of spins increases. However, when L increases from an integer value I to $I + \delta$ ($\delta \ll 1$), the decrease of F is small, then f increases. This behaviour is also observed for small values of x, from $x = 0$ to $x \approx 0.4$.

These oscillations are related to the oscillations of the mean coordination number. Let $B(I, x)$ be the total number of bonds in a very long strip of length N. We can write $B(I, x) = B_i(x) + B_c(I)$, where $B_i(x)$ is the number of bonds between a site in the

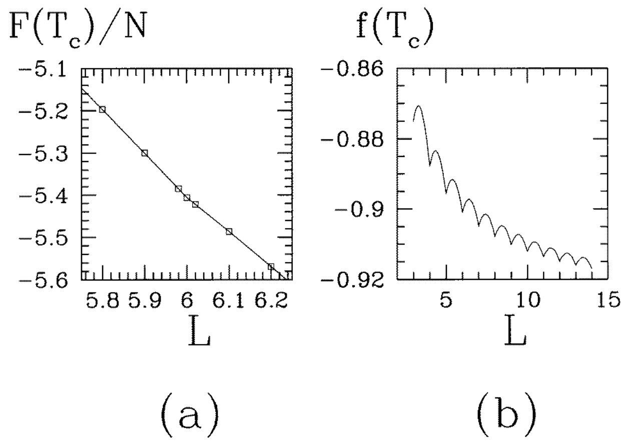

Figure 2. (*a*) Free energy per column at T_c in strips with one incomplete row, for values of L near 6. Lines connecting the data are drawn to guide the eye. (*b*) Free energy per spin at T_c in the same strips, for all values of L analysed.

incomplete row and one of its neighbours (in the incomplete row or in the row below it—figure 1(*a*)), and $B_c(I)$ is the number of bonds between two sites in the complete rows. The probability of finding a cluster of m connected sites in the incomplete row is $x^m(1-x)^2$, we then obtain

$$B_i(x) = Nx + N\sum_{m=2}^{\infty} x^m(1-x)^2(m-1) = N(x+x^2). \tag{2}$$

In equation (2), the first term (Nx) is the number of vertical bonds, and the second term (Nx^2) is the number of horizontal bonds which contribute to $B_i(x)$ (figure 1(*a*)). On the other hand, $B_c(I) = NI + N(I-1) = N(2I-1)$. Then the mean coordination number is

$$q(I,x) = \frac{B(I,x)}{N(I+x)} = 2 + \frac{x^2-x-1}{L}. \tag{3}$$

For fixed I, $q(I,x)$ has a minimum for $x = x_M$, which depends on I. For $4 \leqslant I \leqslant 13$ it is always near 0.4; for instance, for $I = 4$ the minumum is at $x_M \approx 0.359$ and for $I = 13$, at $x_M \approx 0.454$. We also observe that the maximum of f in the range $4 < L < 5$ is located between $x = 0.3$ and $x = 0.4$, while the maximum of f in the range $13 < L < 14$ is located between $x = 0.4$ and $x = 0.5$. It suggests a relation between these oscillatory behaviours.

In figure 3 we show $f + Aq(I,x)$ versus L with $A = 0.145$. The relative amplitude of the oscillations are much smaller than in figure 2(*b*). Higher-order corrections in $q(I,x)$ would probably make that plot smoother. Then the oscillations may be interpreted as surface corrections to the free energy. These corrections are related to $q(I,x)$, which measures the number of absent bonds when compared to a homogeneous lattice ($q = 2$—the two-dimensional system or a strip with periodic boundary conditions, where $1/L$ corrections vanish). Note also that $q(I,x)$ has an $1/L$ dependence (equation (3)), characteristic of surface corrections.

The oscillations of f have interesting consequences to finite-size scaling. Although f is not a monotonic function of continuous L, it scales in a simple form for values of L differring by integers, i.e. $L = I + x$ with variable I and fixed x. This property is clear in figure 4, where we show $f - f_\infty$ versus $1/L$, with $f_\infty = -0.929\,695\,398\ldots$ (the exact

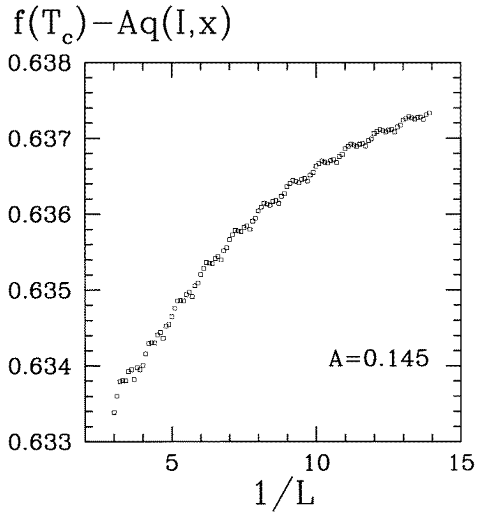

Figure 3. Free energy per spin at T_c with a correction proportional to the mean coordination number $q(I, x)$. The value of A is chosen to minimize the relative amplitudes of the oscillations.

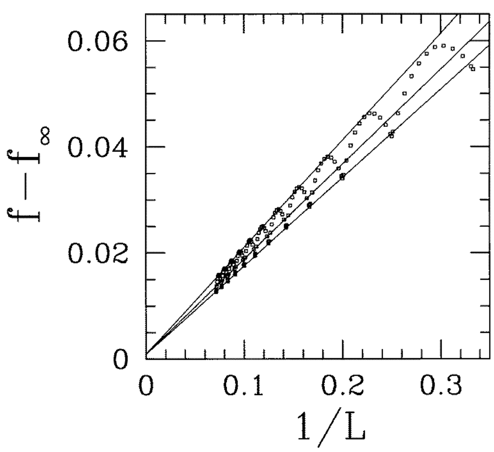

Figure 4. Difference between the free energy per spin of strips with one incomplete row and the free energy per spin of the two-dimensional Ising model [17] at T_c versus $1/L$. Straight lines are least squares fits of the data for $x = 0$ (lower line), $x = 0.1$ and $x = 0.4$ (upper line).

value of the free energy per site at T_c in the square lattice [17]). Straight lines are least squares fits (using $L \geqslant 5$) of three sets of points: $x = 0$, $x = 0.1$ and $x = 0.4$. The convergence of f to f_∞ is good, with errors less than 10^{-3}, and indicate that the amplitude of the oscillations of f decreases approximately as $1/L$. In section 5 we will show that the fits of f by higher-order polynomials lead to accurate estimates of the conformal anomaly $c = \frac{1}{2}$, and we will also discuss the surface corrections of f.

Finally, we observe that the oscillations of f are still present at other temperatures. In figures 5(*a*) and (*b*) we show f versus L at $T = T_c/2$ and $T = 2T_c$. It proves that the behaviour of f is not just a consequence of the specific properties of the model, such as the transition of the two-dimensional system at T_c, but is intimately related to the geometry of the system. The local maxima of f are also at values of x near 0.4, and these values increase with I, similarly to the minima of $q(I, x)$. At those temperatures, however, a first-order correction in $q(I, x)$ does not reduce the oscillations of f as done at T_c (figure 3). Higher-order corrections in $q(I, x)$ must then be necessary for that purpose.

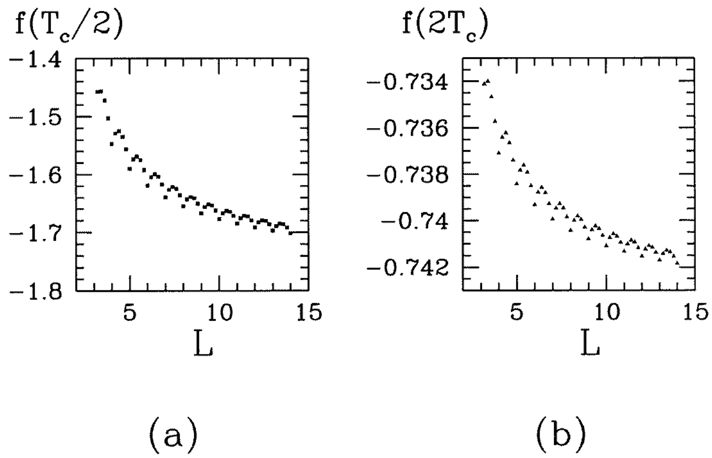

Figure 5. (*a*) Free energy per spin at $T = T_c/2$ versus length L in strips with one incomplete row. (*b*) Free energy per spin at $T = 2T_c$ versus length L in the same strips.

3. Specific heat and magnetic susceptibility of strips with one incomplete row

In order to obtain the specific heat per spin C_L at temperature T, we must calculate the free energy at T, $T_1 = 0.999T$ and $T_2 = 1.001T$, with $h = 0$, and its numerical second-order derivative:

$$C_L = -T\frac{\partial^2 f}{\partial T^2}. \tag{4}$$

In figure 6 we show the specific heat $C_L(T_c)$ versus $\ln L$. C_L also oscillates, with local maxima at integer L. The minima of C_L is also at x near 0.4, and the location of these minima increase with I.

We expect that

$$C_L(T_c) \sim \ln L \tag{5}$$

for integer L [2], and it is confirmed by the linear fit of the data for $L = I \geqslant 7$. This scaling is also obtained with $L = I + x$, for variable I and fixed x.

The initial susceptibility

$$\chi_L = \left(\frac{\partial^2 f_L}{\partial h^2}\right)_{h=0} \tag{6}$$

is obtained numerically from the free energies calculated at $(T, h = 0)$ and $(T, h = 10^{-4})$, with h in units of $J/k_B T$.

In figure 7 we show $\ln\chi_L(T_c)$ versus $\ln L$, and we note that this quantity also oscillates. The decrease of χ_L for small x is shown in greater detail in the inset of figure 7, for $L \approx 7$. However, the susceptibility does not follow the oscillations of $q(I, x)$.

It is expected that $\chi_L(T_c)$ scales as

$$\chi_L(T_c) \sim L^{\gamma/\nu} \tag{7}$$

with $\gamma/\nu = 1.75$, as $L \to \infty$. The linear fit of the data for $L = I \geqslant 7$ in figure 7 is consistent with a value $\gamma/\nu \approx 1.7$. The data for $L = I + x$, with fixed x and variable I, also scale with a ratio γ/ν near this value.

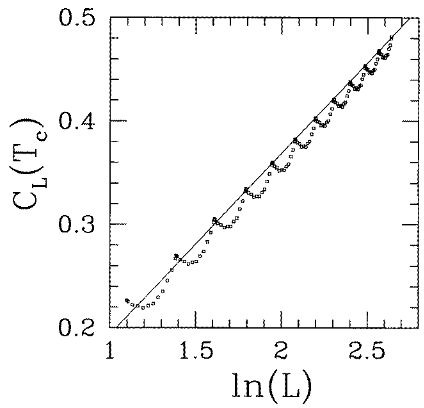

Figure 6. Specific heat per spin at T_c versus length L in strips with one incomplete row. The line is a linear fit of the data for integer $L \geqslant 7$.

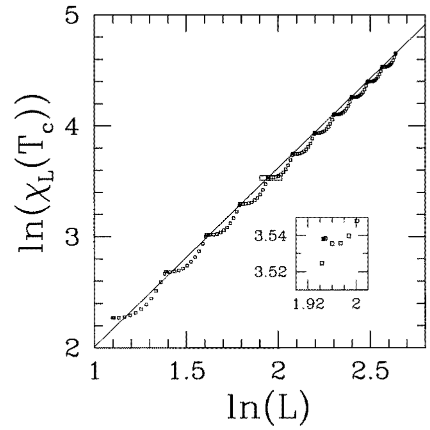

Figure 7. Magnetic susceptibility per spin at T_c and $h = 0$ versus length L in strips with one incomplete row. The line is a linear fit of the data for integer $L \geqslant 7$. The inset shows the decrease of $\chi_L(T_c)$ after $L = 7$ (the full square is the result for $L = 7$).

We also observe that the amplitudes of the oscillations of C_L and χ_L decrease approximately as $1/L$, similarly to the oscillations of the free energy.

The standard method of obtaining two-dimensional critical exponents from calculations in strips is to obtain their finite-size estimates and extrapolate them to $L \to \infty$. In the case of the ratio of exponents γ/ν, the finite-size estimates are

$$\left(\frac{\gamma}{\nu}\right)_{L,\delta} = \frac{\ln[\chi_L/\chi_{L-\delta}]}{\ln[L/(L-\delta)]} \tag{8}$$

and $\delta = 1$ is generally used [2, 8, 14].

Our results suggest that this method may be applied not only to integer L, but also to non-integer L if $\delta = 1$. In figure 8 we show $(\frac{\gamma}{\nu})_{L,1}$ versus $1/L$ (equation (8)). Straight lines are fits of the data for $x = 0$ and $x = 0.5$. They provide estimates of γ/ν near the exact value. Fits to second or third degree curves (not shown in figure 8) provide much better estimates. However, the oscillations of $\chi_L(T_c)$ (figure 7) prove that this method cannot be generalized to non-integer δ in equation (8). Equivalently, equation (7) is not valid for continuous L, but requires oscillatory corrections.

Thus, the main result which we have obtained is that the thermodynamic quantities in strips with a single incomplete row cannot be interpolated by their values in strips with complete rows (integer L). Then, finite-size scaling relations, such as equations (5) and (7), do not apply to continuous L, but are still valid if the values of L differ by integers.

4. Strips with Gaussian distributions of widths

Now we consider strips with discretized Gaussian distributions of widths whose rms deviation is $\Delta L = 1$ (constant for all L). We have calculated the free energy f (equation (1)), the specific heat C_L (equation (4)) and the magnetic susceptibility χ_L (equation (6)), at $T = T_c$ and $h = 0$, using the same techniques presented in sections 2 and 3. We have considered strips of mean widths $3 \leqslant L \leqslant 11$, with interval 0.1 between consecutive L. The free energy at T_c and $h = 0$ was calculated up to $L = 14$, in order to

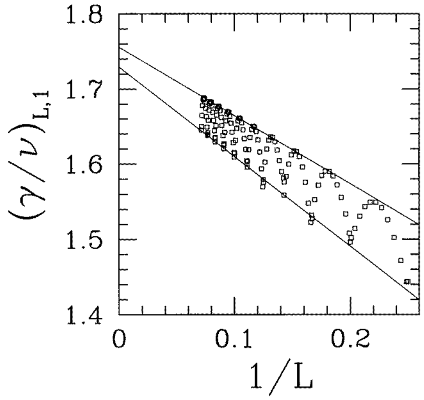

Figure 8. Finite-size estimates of the ratio of exponents γ/ν (equation (10) with $\delta = 1$) in strips with incomplete rows. Lines are linear fits of the data for $x = 0$ (lower line) and $x = 0.5$ (upper line), using $L \geqslant 7$.

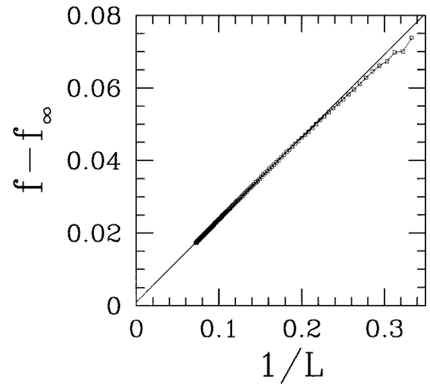

Figure 9. Difference between the free energy per spin of Gaussian strips and the free energy per spin of the two-dimensional Ising model [17] at T_c versus $1/L$. The line is a linear fit of the data for $L \geqslant 5$.

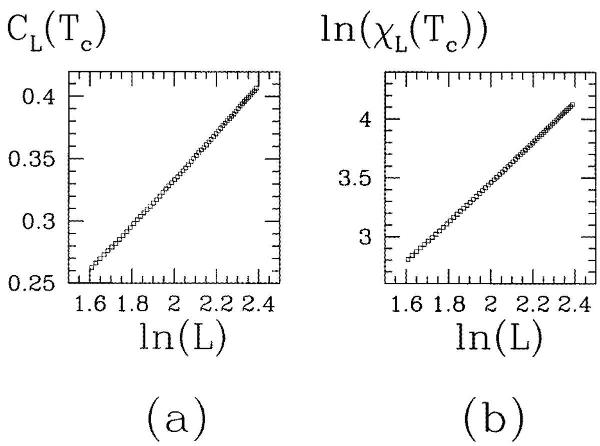

Figure 10. (*a*) Specific heat per spin at T_c versus length L in Gaussian strips with $\Delta L = 1$. (*b*) Magnetic susceptibility per spin at T_c and $h = 0$ versus length L in the same strips.

estimate the conformal anomaly (section 5). The strip lengths were $N = 10^5$.

In figure 9 we show $f - f_\infty$ at T_c versus $1/L$ in the Gaussian strips. We note that the large oscillations of the results in the strips with incomplete rows are absent in this case. The error in the linear fit is nearly 10^{-3}, taking into account the estimates for all $L \geqslant 5$. In figure 10(*a*) we show $C_L(T_c)$ versus $\ln L$ and in figure 10(*b*) we show $\ln(\chi_L(T_c))$ versus $\ln L$ in the Gaussian strips. We also do not observe large oscillations in those plots (small fluctuations in the inclinations are observed in high-resolution plots).

It is interesting to observe that the physical quantities vary monotonically with L in systems which are rough even for integer L. Although the dimensionality of thin magnetic films is different, we devise the possibility of finding similar features in those structures. If the roughness pattern is approximately the same for all mean thicknesses (integer or not), physical quantities must vary monotonically with L, at a fixed temperature. On the other hand, those quantities may oscillate in the case of layer-by-layer growth if there is one

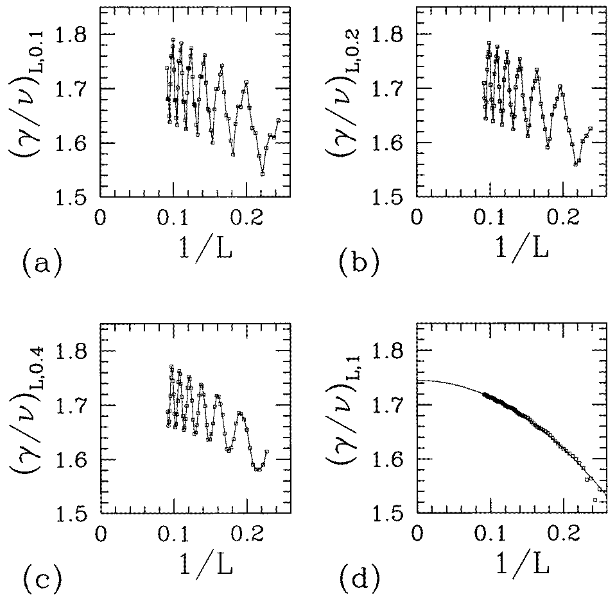

Figure 11. Finite-size estimates of the ratio of exponents γ/ν (equation (10)) in Gaussian strips with $\Delta L = 1$: (*a*) $\delta = 0.1$, (*b*) $\delta = 0.2$, (*c*) $\delta = 0.4$, (*d*) $\delta = 1$. In (*a*)–(*c*), lines connecting the data are drawn to guide the eye. In (*d*) we show a second-degree curve fitting the data for $L \geqslant 7$.

partially filled layer, as we have observed in the strips with a single incomplete row. It is not clear, however, whether this analysis can be extended or not to quantities such as critical temperatures. In order to examine this question, a three-dimensional model should certainly be considered.

Despite the apparently smooth behaviour of f, C_L and χ_L (figures 9, 10(*a*) and (*b*)), finite-size scaling relations with continuous L, such as equations (5) and (7), must be carefully analysed. Linear fits of all the data with $L \geqslant 7$ in figures 10(*a*) and (*b*) provide reasonable estimates of critical exponents ($\alpha = 0$—logarithm—and $\gamma/\nu = 1.75$). This process parallels the experimental methods of estimating critical exponents. On the other hand, more refined techniques of theoretical analysis, such as equation (8), may present some problems. In figures 11(*a*)–(*d*) we show $(\frac{\gamma}{\nu})_{L,\delta}$ versus $1/L$ for $\delta = 0.1$, $\delta = 0.2$, $\delta = 0.4$ and $\delta = 1$. For $\delta = 1$ the trend to $\frac{\gamma}{\nu} = 1.75$ is clear (a fit to a second-degree curve is shown in figure 11(*d*)), but for the other values of δ there are oscillations whose amplitudes do not decrease as L increases. These amplitudes do not diverge when $\delta \to 0$ (note that the amplitudes for $\delta = 0.1$ and $\delta = 0.2$ are nearly the same), which is consistent with the small fluctuations in the inclinations in figure 10(*b*).

The discretization of the Gaussian distributions of widths is responsible for those oscillations. In fact, for integer L, the distribution is dominated by three values of widths ($L - 1$, L and $L + 1$), while for half-integer $L = I + 0.5$ it is dominated by two widths (I and $I + 1$). Thus the increase of the mean width ($L \to L + \delta$ in equation (10)) may have different effects on the susceptibility for integer and half-integer L.

The main conclusion from the results above is that thermodynamic quantities in Gaussian strips with non-integer mean width L can be interpolated, with a reasonable accuracy, by their values in Gaussian strips with integer L (figures 9 and 10). Then, finite-size scaling

Table 1. For strips with one incomplete row ($L = I + x$, fixed x), M is the degree of the polynomial in $1/L$ which gives the best estimate of the critical free energy f_∞, and f_s and Δ are first- and second-order terms of the expansion (equation (8)). c is the conformal anomaly. Errors in the last digits, shown in parentheses, are standard deviations given by the fitting procedure without considering error bars in the data.

x	M	f_∞	f_s	Δ	$c = 24\Delta/\pi$
0.1	7	−0.929 695 40(1)	0.194 726 87(5)	−0.065 445(1)	0.499 96(1)
0.2	6	−0.929 695 37(1)	0.204 819 7(1)	−0.065 417(2)	0.499 75(2)
0.3	6	−0.929 695 38(1)	0.212 502 47(9)	−0.065 424(2)	0.499 80(2)
0.4	6	−0.929 695 38(1)	0.217 292 1(1)	−0.065 431(2)	0.499 86(2)
0.5	6	−0.929 695 38(1)	0.218 656 92(8)	−0.065 428(1)	0.499 83(1)
0.6	5	−0.929 695 38(1)	0.217 359 78(2)	−0.065 4309(2)	0.499 855(2)
0.7	5	−0.929 695 40(1)	0.213 609 18(5)	−0.065 441(1)	0.499 93(1)
0.8	5	−0.929 695 40(1)	0.205 752 35(4)	−0.065 440(1)	0.499 92(1)
0.9	6	−0.929 695 38(1)	0.195 487 2(1)	−0.065 430(2)	0.499 85(2)

relations with continuous L holds approximately. Small corrections, however, are still present, and they are enhanced by the standard theoretical methods of estimating critical exponents when applied to continuous L.

5. Surface free energy and conformal anomaly with random boundaries

In an infinitely long strip of width L, it is expected that the free energy per spin at criticality has the scaling form [18, 19]

$$f = f_\infty + \frac{f_s}{L} + \frac{\Delta}{L^2} + \cdots. \tag{9}$$

In equation (18), $\frac{1}{2} f_s$ is the surface free energy, which depends on the boundary conditions (it vanishes for periodic boundaries), while Δ is a quantity which characterizes the universality class of the system. For free boundaries, $\Delta = -\pi c/24$, where c is the conformal anomaly or the value of the central charge of the Virasoro algebra [18–20]. For the two-dimensional Ising model, $c = \frac{1}{2}$.

In strips with incomplete rows, it is very difficult to expand f in powers of $1/L$ for continuous L. As shown in section 2, first-order corrections to f are oscillatory and related to $q(I, x)$ (equation (3)). Then a numerical analysis of the whole data would have to consider oscillatory corrections, and a functional form for these corrections (for instance, powers of $q(I, x)$) would have to be assumed. On the other hand, if we analyse separately the data for each value of x, we can expand f in powers of $1/L$ and estimate the corrections of equation (9) with good accuracy.

We have used polynomials of various degrees, and have obtained estimates of f_∞ (zeroth-order term), f_s (first-order term) and Δ (second-order term). Since the exact value $f_\infty = -0.929\,695\,398\ldots$ is known, we considered the estimates from the polynomials which give the best estimates of f_∞. In table 1 we show, for each value of x, the degree M of that polynomial, the respective estimates of f_∞, f_s and Δ, and the corresponding estimate of the conformal anomaly $c = -24\Delta/\pi$.

Our results indicate that $c = \frac{1}{2}$. Moreover, we observe that the surface free energy for $L = I + x$ and $L = I - x$ is nearly the same, since the boundaries are very similar: the magnetic sites in the last row of the strip with $L = I + x$ are the non-magnetic (absent) sites in the last row of the strip with $L = I - x$, and vice versa. In fact, when $I \to \infty$,

Table 2. For Gaussian strips ($L = I + x$, fixed x), M is the degree of the polynomial in $1/L$ which gives the best estimate of the critical free energy f_∞, and f_s and Δ are first- and second-order terms of the expansion (equation (8)). c is the conformal anomaly. The last line shows estimates considering all values of L ($5 \leqslant L \leqslant 14$). Errors in the last digits, shown in parentheses, are standard deviations given by the fitting procedure without considering error bars in the data.

x	M	f_∞	f_s	Δ	$c = 24\Delta/\pi$
0	5	−0.929 695 38(1)	0.243 955 91(6)	−0.065 429(1)	0.499 84(1)
0.1	5	−0.929 695 38(1)	0.244 091 53(9)	−0.065 427(1)	0.499 83(1)
0.2	5	−0.929 695 38(1)	0.244 363 32(7)	−0.065 431(1)	0.499 86(1)
0.3	5	−0.929 695 38(1)	0.244 580 67(5)	−0.065 429(1)	0.499 84(1)
0.4	5	−0.929 695 37(1)	0.244 345 6(1)	−0.065 424(2)	0.499 80(2)
0.5	5	−0.929 695 38(1)	0.244 760 68(8)	−0.065 426(1)	0.499 82(1)
0.6	5	−0.929 695 38(1)	0.244 412 46(7)	−0.065 426(1)	0.499 82(1)
0.7	5	−0.929 695 38(1)	0.244 137 89(7)	−0.065 430(1)	0.499 85(1)
0.8	5	−0.929 695 38(1)	0.243 751 06(6)	−0.065 429(1)	0.499 84(1)
0.9	5	−0.929 695 38(1)	0.244 095 7(1)	−0.065 428(2)	0.499 83(2)
All	2	−0.929 71(5)	0.244 6(8)	−0.067 4(31)	0.515(24)

$q(I, x)$ is symmetric around $x_M = 0.5$ (equation (3)), which explains the symmetry of f_s around $x = 0.5$.

We have performed the same analysis in the Gaussian strips. In table 2 we show the degree M of the polynomials which give the best estimates of f_∞ and the corresponding estimates of f_s, Δ and c, for each value of x (considering $L = I + x$). Only data with $L \geqslant 5$ were considered. The results are also consistent with a universal value $c = \frac{1}{2}$, and the surface free energy is nearly the same for all values of x. The same quantities were estimated using all values of L together (from 5 to 14, at intervals 0.1), and are shown in the last line of table 2. In that case, the errors are much larger, but still consistent with $c = \frac{1}{2}$. This problem is certainly related to the discretization procedure, which leads to slightly different boundary effects for different x, as discussed in section 4.

Thus, the main conclusion is that the conformal anomaly $c = \frac{1}{2}$ is obtained, with good accuracy, in the case of random boundaries, for all types of randomness considered here.

6. Summary and conclusion

We studied the Ising model on two types of two-dimensional finite structures with non-integer characteristic lengths.

First we considered strips with I complete rows and one partially filled row with probability x (length $L = I + x$). In those strips, the free energy, the specific heat and the magnetic susceptibility per spin oscillate as L increases, at fixed temperature. This behaviour is related to the oscillations in the mean coordination number of the strip, while the last row is being filled. The main consequence of those oscillations is that finite-size scaling cannot be generalized to continuous L unless oscillatory corrections are considered. The amplitude of these corrections decay as $1/L$, but they are still very high for $L \approx 10$. On the other hand, for values of L differing by integers ($L = I + x$, with variable I and fixed x), finite-size scaling relations are satisfied, with the usual corrections in powers of $1/L$.

We also considered strips with discretized Gaussian distributions of widths (mean width L and rms deviation $\Delta L = 1$). In these structures, there are no oscillations of the thermodynamic quantities as functions of continuous L. Small fluctuations in declivities, which are detected in the calculation of critical exponents, are related to the discretization of the width distribution. Thus finite-size scaling relations are satisfied with reasonable accuracy for continuous L.

The fits of the free energy as a function of $1/L$ at T_c indicate that the conformal anomaly is $c = \frac{1}{2}$ for all types of random boundaries considered here.

If thermodynamic quantities vary smoothly with L in magnetic systems with non-integer characteristic lengths, then our results suggest that they are not grown layer-by-layer, but instead that their roughness pattern is constant for all L (or, possibly, they have another roughness pattern with a smooth relation between L and ΔL). It would be interesting to pursue the investigation of these features both in real systems (e.g. thin magnetic films) and in three-dimensional models.

References

[1] Fisher M E 1971 Critical phenomena *Proc. Int. School of Physics 'Enrico Fermi' (Course LI, Varenna, 1970)* ed M S Green (New York: Academic)
[2] Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic)
[3] Heinrich B and Cochran J F 1993 *Adv. Phys.* **42** 523
[4] Merikoski J, Timonen J, Manninen M and Jena P 1991 *Phys. Rev. Lett.* **66** 938
[5] Elmers H J, Hauschild J, Höche H, Gradmann U, Bethge H, Heuer D and Köhler U 1994 *Phys. Rev. Lett.* **73** 898
[6] Garreau G, Farle M, Beaurepaire E and Baberschke B 1997 *Phys. Rev.* B **55** 330
[7] Stindtmann M, Farle M, Rahman T S, Benabid L and Baberschke K 1997 *Surf. Sci.* **381** 12
[8] Aarão Reis F D A 1997 *Phys. Rev.* B **55** 11 084
[9] Aarão Reis F D A 1998 *Phys. Rev.* B **58** 394
[10] Aarão Reis F D A 1998 Physica A **257** 495
[11] Nightingale M P 1990 *Finite Size Scaling and Numerical Simulations of Statistical Systems* ed V Privman (Singapore: World Scientific)
[12] Blöte H W J and Nightingale M P 1982 *Physica* A **112** 405
[13] Glaus U 1987 *J. Phys. A: Math. Gen.* **20** L595
[14] Aarão Reis F D A, de Queiroz S L and dos Santos R R 1997 *Phys. Rev.* B **56** 6013
[15] Sen P, Stauffer D and Gradmann U 1997 *Physica* A **245** 361
[16] Onsager L 1944 *Phys. Rev.* **65** 117
[17] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
[18] Blöte H W, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
[19] Affleck I 1986 *Phys. Rev. Lett.* **56** 746
[20] Virasoro M A 1970 *Phys. Rev.* D **1** 2933